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GAUSS ON INFINITY

BY WILLIAM C. WATERHOUSE
THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PA 16802

SUMMARIES

In opposing the use of completed infinity in mathematics, Gauss was making a valid criticism of one particular kind of argument. His celebrated statement has no connection with the set theory to which it was later applied.

Gauss cum dixisset quantitates infinitas geometris esse evitandas, de errore quodam certo recte monuit. Sententia haec notissima ad doctrinam acervorum minime pertinet.

L'opposition de Gauss à l'emploi des quantités infinies provenait d'une erreur géométrique incontestable, et n'avait rien à faire avec la théorie des ensembles.

1. INTRODUCTION

In a famous sentence Gauss once wrote, "I protest against the use of an infinite quantity as something completed, which is never permissible in mathematics." Kline's widely read history quotes this passage in the section headed "The Concept of an Infinite Set," where the previous paragraph deals with Galileo's paradox on infinite sets, and the following sentence begins, "Cauchy, like others before him, denied the existence of infinite sets..." [Kline 1972, 993]. Similarly, a substantial recent paper on Cantor assumes without question that the words imply a full-fledged finitism:

> *... 3 is not as close to the true value of π as is 3.14, and 3.14159 is still closer. By adding additional places to the right of the decimal, it is possible to approximate the true value of π as closely as one likes. But Gauss insisted that one could not assume all the terms of the decimal expansion to be given to determine π exactly. To do so would involve an infinite number of terms, and thus comprise an actually infinite set of numbers, which Gauss refused to allow in rigorous mathematics* [Dauben 1977, 86].

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Interpretations like these seem to be fairly common. But actually, as we shall see, such a meaning was not intended by Gauss and only became attached to his words half a century later.

2. SCHUMACHER ON THE PARALLEL POSTULATE

The story begins with the following proof of the parallel postulate. Recall first that it is enough to show that the sum of the angles of a triangle is 180°. Now extend the sides of a triangle *ABC* [Fig. 1], and choose a radius so large that in relation to it the sides are as small as you like. With this radius draw around *C* the semicircle *DEFG*. The arc *DE* then measures the angle *DCE*. But *AC* is negligible compared with the radius, so *A* and *C* essentially coincide, and the arc *FG* measures the angle *FAG*. Similarly *EF* measures *EBF* = *ABC*. Thus the sum of the angles is 180°; or, to be precise, we have shown that it differs from 180° by less than an arbitrarily small amount.

This proof was sent to Gauss by his friend H. Schumacher in 1831. The next letter from Gauss made no mention of it, and so Schumacher wrote specifically asking for an opinion:

> *You can easily imagine that your judgment is very important to me, since you so easily discover every weakness in a proof.... If anyone should think (though I do not) that a proof is needed for the proposition that one can consider the vertices of a triangle as coinciding centers of a circle of infinite radius ..., this proof can be rigorously carried through* [Gauss 1900, 214].

3. GAUSS ON SCHUMACHER'S PROOF

Gauss replied at once. To simplify the discussion, he reduced the argument to its basic point [Fig.2]:

(1) In a triangle where one side is finite but the second and third are infinite, the two angles at the finite side add to 180°.

He restated Schumacher's proof in this case, showing how the

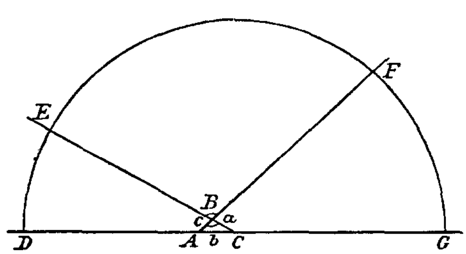

FIGURE 1

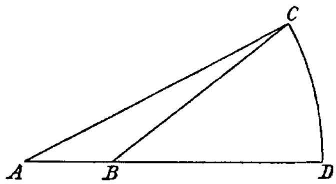

FIGURE 2

general result followed from it, and continued:

> *As for your proof of (1), I protest first of all against the use of an infinite quantity as a* completed *one, which is never permissible in mathematics. The infinite is only a* façon de parler, *where one is really speaking of limits to which certain ratios come as close as one likes while others are allowed to grow without restriction. In this sense Non-Euclidean geometry contains nothing at all contradictory. No doubt people must think many of its results paradoxical at first, but to think this contradictory would be only self-deception....*
>
> *In the case in question, now, when the points* A, B *and the direction* AC *are given and* C *can grow without bound, there is nothing at all contradictory in saying that, though angle* DBC *comes closer and closer to* DAC, *yet the difference can never be reduced below a definite finite amount* [Gauss 1900, 216-217].

4. GAUSS ON INFINITY

Here, in context, we have Gauss' famous statement. Schumacher in his reply said he did not think he had assumed any actual infinity, and indeed none appears in his proof. But the second letter shows that he believed the crucial step because he thought of it intuitively as saying that the vertices of a triangle were coinciding centers of a circle of infinite radius. Thus Gauss' remark is quite to the point. There is no harm in using the intuitively appealing language of behavior at infinity, but one must be prepared to demonstrate in detail the statements thus abbreviated, since they may not be as "obvious" as they seem. At the very least, as Schumacher's proof shows, they may involve nontrivial assumptions.

Gauss had understood this long before. In his thesis (1799), for instance, his idea is essentially to write $f(x + iy) = P(x, y) + iQ(x, y)$ and prove the fundamental theorem of algebra by showing that the algebraic plane curves $P(x, y) = 0$ and $Q(x, y) = 0$ must intersect. At one point, working in polar coordinates r, ϕ around an origin C, he writes:

> *At an infinite distance from the point* C *the*

first curve, with equation

$$\sin m\phi + \frac{A}{r}\sin(m-1)\phi + \frac{B}{rr}\sin(m-2)\phi \text{ etc.} = 0,$$

coincides with that curve whose equation is $\sin m\phi = 0$ ... *The first curve therefore has 2m infinite branches* ... [Struik 1969, 117-118].

This of course is perfectly satisfactory if one is sure of what it means. But since the thesis began with a criticism of other attempted proofs, Gauss does not want to leave any doubt, and so, since "some readers might be offended by infinitely large quantities," he proceeds to translate the argument explicitly into a collection of estimates and inequalities.

As we see again here, when Gauss spoke of infinity in mathematics, he referred specifically to the analytic/geometric intuition of behavior at infinity. He did not object to the use of such language. Later in the letter to Schumacher, for instance, he takes a formula about circumferences of circles in Non-Euclidean geometry and restates it "in the metaphorical language of infinity"; and as we saw, Schumacher's original proof did not even mention infinity explicitly. Gauss' objection reduces precisely to this: here as elsewhere, geometric intuition is no substitute for proof. In establishing results in ordinary geometry or analysis, one is not entitled casually to assume statements about behavior at infinity. Needless to say, this is not a controversial statement; Gauss is (as usual) unquestionably right.

5. OTHERS ON INFINITY

Not everyone perhaps saw the point as clearly as Gauss, but it was hardly a new idea in his time. Careful students of the calculus had been saying similar things for many years, about infinity as well as about infinitesimals. The famous early examples are in the articles by D'Alembert in the *Encyclopédie*: "D'Alembert said that ... the differential notation is to be considered merely as a convenient abridgment or manner of speaking, used to avoid the circumlocution necessary in expressing the limit concept." Similarly, "he asserted ... that the notion of infinity ... is only a convenient abridgment for the interpretation in terms of the doctrine of limits" [Boyer 1949, 248-249]. A decade before 1831 this development had already reached its apex in Cauchy's *Cours d'Analyse,* where infinitesimal x and infinite x were explicitly defined to mean variable x approaching zero or growing beyond all bounds [Cauchy 1897, 37-38]. Nor was the idea limited to France; it appeared (with mild confusion) in a popular German textbook published in 1761 by A. G. Kästner [Boyer 1949, 250]. We may recall that Kästner was the mathematics professor at Göttingen; he was almost eighty when Gauss came there as a student, and Gauss (to put it mildly) did not consider him an outstanding mathematician (see e.g. [Kline 1972, 754]). Thus when Gauss called infinity a manner of speaking about limits, he probably felt he was repeating a truism, and Schumacher accepted it without objection.

Since that time, other mathematical concepts have been introduced bearing the name "infinity." Abraham Robinson, for instance, defined infinite and infinitesimal numbers as certain elements of "nonstandard models," proving that such models exist and that certain conclusions can be transferred from them to ordinary numbers [Robinson 1966]. This to be sure is still demonstrably equivalent to a theory of limits. But projective geometers introduced a purely algebraic definition using homogeneous coordinates, and it leads to points at infinity even over fields where no ordinary notion of limits exists. Yet no one, I suspect, will be tempted to cite Gauss as an opponent of homogeneous coordinates: their validity is not at all controversial, and it would be plain that Gauss had no such thing in mind. By examining his discussion of "infinity," we have seen that it likewise had for him no connection with set theory. But the controversy arising in that area led to misinterpretation.

6. CANTOR ON INFINITY

Several different mathematical senses of "infinity" were known to Cantor; in one paper he discusses points at infinity in function theory and variables growing without bound, as well as his own "infinite numbers" [Cantor 1932, 165-166]. But "infinity" also meant much more to him. It must be remembered that the phrase "actual infinity," arising from the general Aristotelian distinction of "potential" and "actual," was not basically a technical term in mathematics. It had of course been used in discussions of the foundations of the calculus. But it had also had a flourishing career in medieval theology and philosophy, and philosophy and religion played a major role in Cantor's later writings on infinity.

The reason for this is partly psychological. In mathematics, Cantor's set theory was under attack as a questionable new subject. By putting his work in a philosophical context, he could consider it as a contribution to a long-standing debate. "The existence of the actually infinite," though often denied, had at least been debated by metaphysicians for centuries, and Cantor's work indeed attracted the interest of several theologians and philosophers [Dauben 1977]. As Meschkowski writes, Cantor "saw in his theory a modern confirmation of scholastic theories on the actually infinite. It is understandable that the investigator, criticized by many of his contemporaries, sought confirmation where it could be found: in discussion with Catholic theologicans" [Meschkowski 1965, 510]. Moreover, the religious feeling was by no means absent; Cantor was quite ready to say that opponents of infinity suffered from a "shortsightedness that robs them of the possibility of seeing the actually infinite, though in its highest, absolute bearer it created and preserves us..." [Cantor 1932, 374-275]. He was also prepared to defend "actual infinity" metaphysically, believing that his set theory was a valid mathematical interpretation of it.

7. CANTOR ON GAUSS

References to Gauss in connection with set theory apparently all go back to a comment published by Cantor himself. In an article of 1885 (published 1886) he wrote:

> *It is just two years ago that Herr Rudolf Lipschitz in Bonn drew my attention to a certain place in the Gauss-Schumacher correspondence where the former speaks out against* any *introduction of actual infinity into mathematics (Letter of 12 July 1831). I answered thoroughly, and* on this point *did not accept the authority of Gauss, which I respect so highly in all other areas..."* [Cantor 1932, 371].

This in isolation makes it seem that Cantor considered Gauss an opponent of his set theory. But as one follows his discussion it becomes clear that this disagreement is only with the words, not with Gauss' actual ideas. The crucial sentence is this:

> ... *a justified antipathy to such* illegitimate *actual infinity has produced in broad areas of science, under the influence of the modern epicurean-materialistic trend, a certain Horror Infiniti which discovers its classical expression and support in the cited writing of Gauss...* [Cantor 1932, 374].

Thus Cantor objected not to Gauss' statement in context but to the meaning attributed to it by his own contemporaries. He had a new mathematical theory with which he could reasonably associate the term "actual infinity" with all its philosophical implications and religious resonance. In this sense Cantor was ready to defend "actual infinity in mathematics" against anyone, even against someone quoting Gauss. He knew, however, that Gauss had been referring only to an "illegitimate" infinity standing for limits, and he had already explained that his infinite numbers had "nothing at all in common with" such ideas [Cantor 1932, 166].

8. CONCLUSIONS

The term "actual infinity" comes from philosophy, and does not refer specifically to any single mathematical idea. When Gauss opposed the use of a completed infinity in mathematics, he was employing language familiar at his time to refer to a particular kind of argument. The example prompting his statement illustrates why his opinion on the point in question is now universally accepted. However, since the philosophical term was ambiguous, Cantor later could equally well apply it to his own mathematical results, and this led Lipschitz to interpret Gauss' words as a Kroneckerian opposition to set theory. As A. Fraenkel carefully put it, Gauss' authority "contributed to his remark being followed in an unrestricted sense" [Fraenkel 1966, 1]. There is no indication that Gauss himself intended such a meaning.

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